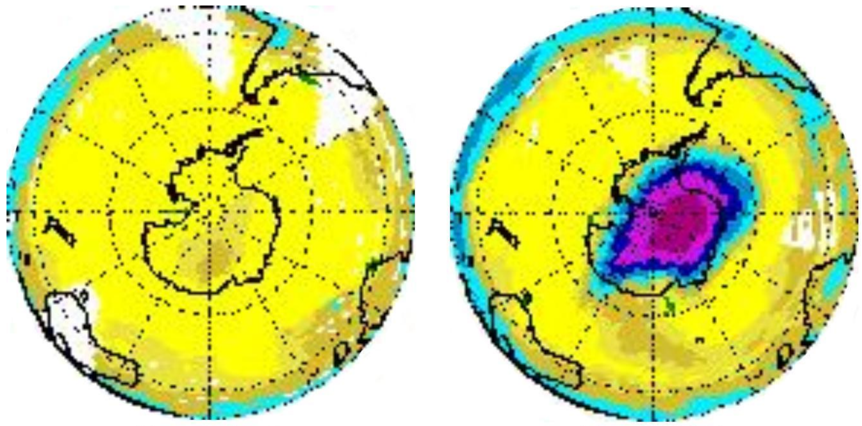


In recent decades, a huge hole in the ozone layer has spread out from Antarctica.

Chapter 34
Light as a particle

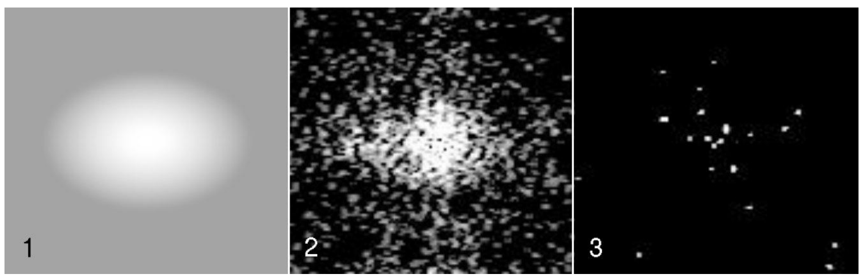
> The only thing that interferes with my learning is my education.
>
> *Albert Einstein*

Radioactivity is random, but do the laws of physics exhibit randomness in other contexts besides radioactivity? Yes. Radioactive decay was just a good playpen to get us started with concepts of randomness, because all atoms of a given isotope are identical. By stocking the playpen with an unlimited supply of identical atom-toys, nature helped us to realize that their future behavior could be different regardless of their original identicality. We are now ready to leave the playpen, and see how randomness fits into the structure of physics at the most fundamental level.

The laws of physics describe light and matter, and the quantum revolution rewrote both descriptions. Radioactivity was a good example of matter's behaving in a way that was inconsistent with classical physics, but if we want to get under the hood and understand how nonclassical things happen, it will be easier to focus on light rather than matter. A radioactive atom such as uranium-235 is after all an extremely complex system, consisting of 92 protons, 143 neutrons, and 92 electrons. Light, however, can be a simple sine wave.

However successful the classical wave theory of light had been — allowing the creation of radio and radar, for example — it still failed to describe many important phenomena. An example that is currently of great interest is the way the ozone layer protects us from the dangerous short-wavelength ultraviolet part of the sun's spectrum. In the classical description, light is a wave. When a wave passes into and back out of a medium, its frequency is unchanged, and although its wavelength is altered while it is in the medium, it returns to its original value when the wave reemerges. Luckily for us, this is not at all what ultraviolet light does when it passes through the ozone layer, or the layer would offer no protection at all!

34.1 Evidence for light as a particle



a / Images made by a digital camera. In each successive image, the dim spot of light has been made even dimmer.

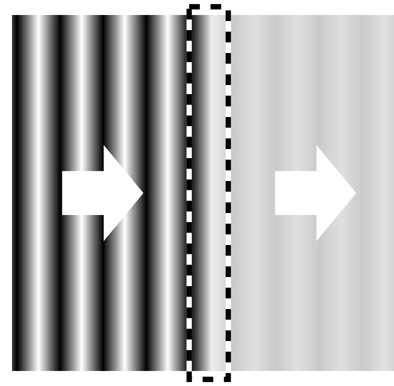
For a long time, physicists tried to explain away the problems with the classical theory of light as arising from an imperfect understanding of atoms and the interaction of light with individual atoms and molecules. The ozone paradox, for example, could have been attributed to the incorrect assumption that the ozone layer was a smooth, continuous substance, when in reality it was made of individual ozone molecules. It wasn't until 1905 that Albert Einstein threw down the gauntlet, proposing that the problem had nothing to do with the details of light's interaction with atoms and everything to do with the fundamental nature of light itself.

In those days the data were sketchy, the ideas vague, and the experiments difficult to interpret; it took a genius like Einstein to cut through the thicket of confusion and find a simple solution. Today, however, we can get right to the heart of the matter with a piece of ordinary consumer electronics, the digital camera. Instead of film, a digital camera has a computer chip with its surface divided up into a grid of light-sensitive squares, called "pixels." Compared to a grain of the silver compound used to make regular photographic film, a digital camera pixel is activated by an amount of light energy orders of magnitude smaller. We can learn something new about light by using a digital camera to detect smaller and smaller amounts of light, as shown in figures a/1 through a/3. Figure 1 is fake, but 2 and 3 are real digital-camera images made by Prof. Lyman Page of Princeton University as a classroom demonstration. Figure 1 is what we would see if we used the digital camera to take a picture of a fairly dim source of light. In figures 2 and 3, the intensity of the light was drastically reduced by inserting semitransparent absorbers like the tinted plastic used in sunglasses. Going from 1 to 2 to 3, more and more light energy is being thrown away by the absorbers.

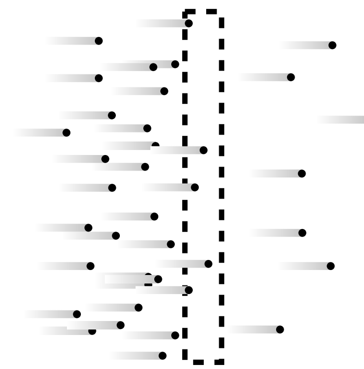
The results are dramatically different from what we would expect based on the wave theory of light. If light was a wave and nothing but a wave, b, then the absorbers would simply cut down the wave's amplitude across the whole wavefront. The digital camera's entire chip would be illuminated uniformly, and weakening the wave with an absorber would just mean that every pixel would take a long time to soak up enough energy to register a signal.

But figures a/2 and a/3 show that some pixels take strong hits while others pick up no energy at all. Instead of the wave picture, the image that is naturally evoked by the data is something more like a hail of bullets from a machine gun, c. Each "bullet" of light apparently carries only a tiny amount of energy, which is why detecting them individually requires a sensitive digital camera rather than an eye or a piece of film.

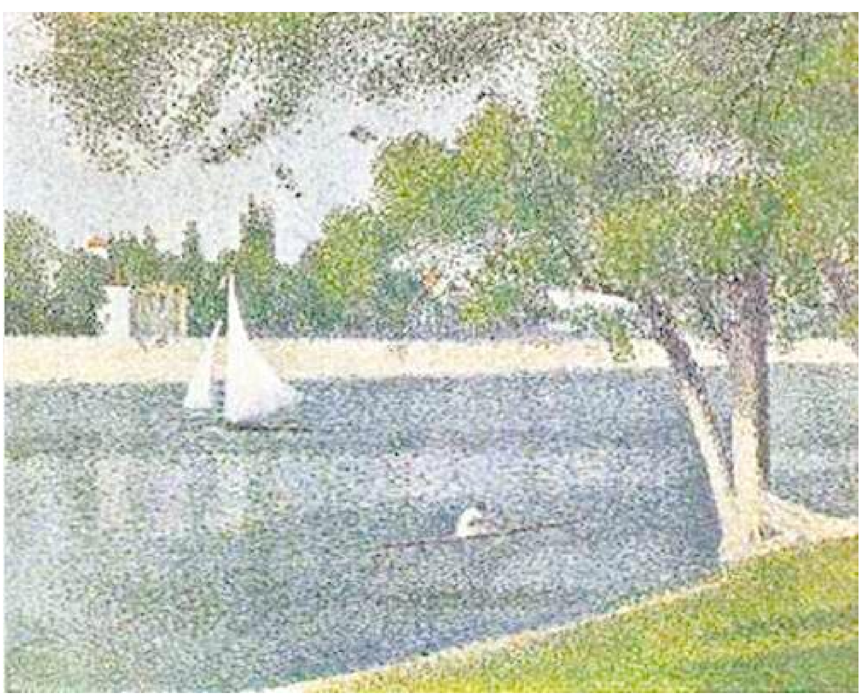
Although Einstein was interpreting different observations, this is the conclusion he reached in his 1905 paper: that the pure wave theory of light is an oversimplification, and that the energy of a beam of light comes in finite chunks rather than being spread smoothly throughout a region of space.



b / A water wave is partially absorbed.



c / A stream of bullets is partially absorbed.



d / Einstein and Seurat: twins separated at birth? Detail from *Seine Grande Jatte* by Georges Seurat, 1886.

We now think of these chunks as particles of light, and call them "photons," although Einstein avoided the word "particle," and the word "photon" was invented later. Regardless of words, the trouble was that waves and particles seemed like inconsistent categories. The reaction to Einstein's paper could be kindly described as vigorously skeptical. Even twenty years later, Einstein wrote, "There are therefore now two theories of light, both indispensable, and — as one must admit today despite twenty years of tremendous effort on the part of theoretical physicists — without any logical connection." In the remainder of this chapter we will learn how the seeming paradox was eventually resolved.

Discussion questions

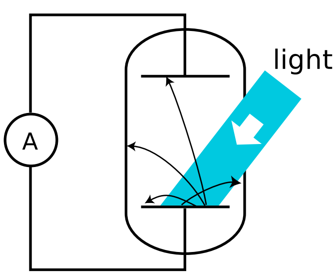
A Suppose someone rebuts the digital camera data in figure a, claiming that the random pattern of dots occurs not because of anything fundamental about the nature of light but simply because the camera's pixels are not all exactly the same — some are just more sensitive than others. How could we test this interpretation?

B Discuss how the correspondence principle applies to the observations and concepts discussed in this section.

34.2 How much light is one photon?

The photoelectric effect

We have seen evidence that light energy comes in little chunks, so the next question to be asked is naturally how much energy is in one chunk. The most straightforward experimental avenue for addressing this question is a phenomenon known as the photoelectric effect. The photoelectric effect occurs when a photon strikes the surface of a solid object and knocks out an electron. It occurs continually all around you. It is happening right now at the surface of your skin and on the paper or computer screen from which you are reading these words. It does not ordinarily lead to any observable electrical effect, however, because on the average, free electrons are wandering back in just as frequently as they are being ejected. (If an object did somehow lose a significant number of electrons, its growing net positive charge would begin attracting the electrons back more and more strongly.)



e / Apparatus for observing the photoelectric effect. A beam of light strikes a capacitor plate inside a vacuum tube, and electrons are ejected (black arrows).

Figure e shows a practical method for detecting the photoelectric effect. Two very clean parallel metal plates (the electrodes of a capacitor) are sealed inside a vacuum tube, and only one plate is exposed to light. Because there is a good vacuum between the plates, any ejected electron that happens to be headed in the right direction will almost certainly reach the other capacitor plate without colliding with any air molecules.

The illuminated (bottom) plate is left with a net positive charge, and the unilluminated (top) plate acquires a negative charge from the electrons deposited on it. There is thus an electric field between the plates, and it is because of this field that the electrons' paths are curved, as shown in the diagram. However, since vacuum is a good insulator, any electrons that reach the top plate are prevented from responding to the electrical attraction by jumping back across the gap. Instead they are forced to make their way around the circuit, passing through an ammeter. The ammeter measures the strength of the photoelectric effect.

An unexpected dependence on frequency

The photoelectric effect was discovered serendipitously by Heinrich Hertz in 1887, as he was experimenting with radio waves. He was not particularly interested in the phenomenon, but he did notice that the effect was produced strongly by ultraviolet light and more weakly by lower frequencies. Light whose frequency was lower than a certain critical value did not eject any electrons at all.[1] This dependence on frequency didn't make any sense in terms of the classical wave theory of light. A light wave consists of electric and magnetic fields. The stronger the fields, i.e., the greater the wave's amplitude, the greater the forces that would be exerted on electrons that found themselves bathed in the light. It should have been amplitude (brightness) that was relevant, not frequency. The dependence on frequency not only proves that the wave model of light needs modifying, but with the proper interpretation it allows us to determine how much energy is in one photon, and it also leads to a connection between the wave and particle models that we need in order to reconcile them.

f / The hamster in her hamster ball is like an electron emerging from the metal (tiled kitchen floor) into the surrounding vacuum (wood floor). The wood floor is higher than the tiled floor, so as she rolls up the step, the hamster will lose a certain amount of kinetic energy, analogous to E_s. If her kinetic energy is too small, she won't even make it up the step.

To make any progress, we need to consider the physical process by which a photon would eject an electron from the metal electrode. A metal contains electrons that are free to move around. Ordinarily, in the interior of the metal, such an electron feels attractive forces from atoms in every direction around it. The forces cancel out. But if the electron happens to find itself at the surface of the metal, the attraction from the interior side is not balanced out by any attraction from outside. In popping out through the surface the electron therefore loses some amount of energy E_s, which depends on the type of metal used.

Suppose a photon strikes an electron, annihilating itself and giving up all its energy to the electron.[2] The electron will (1) lose kinetic energy through collisions with other electrons as it plows through the metal on its way to the surface; (2) lose an amount of kinetic energy equal to E_s as it emerges through the surface; and (3) lose more energy on its way across the gap between the plates, due to

[1] In fact this was all prior to Thomson's discovery of the electron, so Hertz would not have described the effect in terms of electrons — we are discussing everything with the benefit of hindsight.

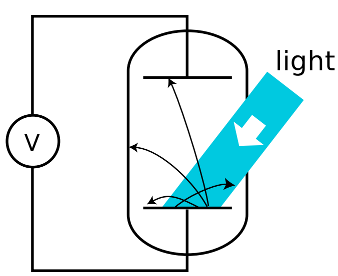
[2] We now know that this is what always happens in the photoelectric effect, although it had not yet been established in 1905 whether or not the photon was completely annihilated.

the electric field between the plates. Even if the electron happens to be right at the surface of the metal when it absorbs the photon, and even if the electric field between the plates has not yet built up very much, E_s is the bare minimum amount of energy that the electron must receive from the photon if it is to contribute to a measurable current. The reason for using very clean electrodes is to minimize E_s and make it have a definite value characteristic of the metal surface, not a mixture of values due to the various types of dirt and crud that are present in tiny amounts on all surfaces in everyday life.

We can now interpret the frequency dependence of the photoelectric effect in a simple way: apparently the amount of energy possessed by a photon is related to its frequency. A low-frequency red or infrared photon has an energy less than E_s, so a beam of them will not produce any current. A high-frequency blue or violet photon, on the other hand, packs enough of a punch to allow an electron to get out of the electrode. At frequencies higher than the minimum, the photoelectric current continues to increase with the frequency of the light because of effects (1) and (3).

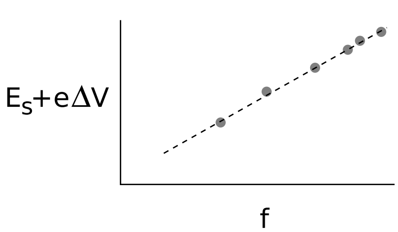
Numerical relationship between energy and frequency

Prompted by Einstein's photon paper, Robert Millikan (whom we encountered in ch. 26) figured out how to use the photoelectric effect to probe precisely the link between frequency and photon energy. Rather than going into the historical details of Millikan's actual experiments (a lengthy experimental program that occupied a large part of his professional career) we will describe a simple version, shown in figure g, that is used sometimes in college laboratory courses. The idea is simply to illuminate one plate of the vacuum tube with light of a single wavelength and monitor the voltage difference between the two plates as they charge up. Since the resistance of a voltmeter is very high (much higher than the resistance of an ammeter), we can assume to a good approximation that electrons reaching the top plate are stuck there permanently, so the voltage will keep on increasing for as long as electrons are making it across the vacuum tube.



g / A different way of studying the photoelectric effect.

At a moment when the voltage difference has a reached a value ΔV, the minimum energy required by an electron to make it out of the bottom plate and across the gap to the other plate is $E_s + e\Delta$V. As ΔV increases, we eventually reach a point at which $E_s + e\Delta V$ equals the energy of one photon. No more electrons can cross the gap, and the reading on the voltmeter stops rising. The quantity $E_s+e\Delta V$ now tells us the energy of one photon. If we determine this energy for a variety of frequencies, h, we find the following simple relationship between the energy of a photon and the frequency of the light:



h / The quantity $E_s + e\Delta V$ indicates the energy of one photon. It is found to be proportional to the frequency of the light.

$$E = hf \qquad ,$$

where h is a constant with a numerical value of 6.63×10^{-34} J·s.

Note how the equation brings the wave and particle models of light under the same roof: the left side is the energy of one *particle* of light, while the right side is the frequency of the same light, interpreted as a *wave*. The constant h is known as Planck's constant (see historical note on page 945).

self-check A

How would you extract h from the graph in figure h? What if you didn't even know E_s in advance, and could only graph $e\Delta V$ versus f? ▷ Answer, p. 1011

Since the energy of a photon is hf, a beam of light can only have energies of hf, $2hf$, $3hf$, etc. Its energy is quantized — there is no such thing as a fraction of a photon. Quantum physics gets its name from the fact that it quantizes things like energy, momentum, and angular momentum that had previously been thought to be smooth, continuous and infinitely divisible.

Number of photons emitted by a lightbulb per second *example 1*

▷ Roughly how many photons are emitted by a 100-W lightbulb in 1 second?

▷ People tend to remember wavelengths rather than frequencies for visible light. The bulb emits photons with a range of frequencies and wavelengths, but let's take 600 nm as a typical wavelength for purposes of estimation. The energy of a single photon is

$$E_{photon} = hf = \frac{hc}{\lambda}$$

A power of 100 W means 100 joules per second, so the number of photons is

$$\frac{100\text{ J}}{E_{photon}} = \frac{100\text{ J}}{hc/\lambda} \approx 3\times10^{20} \quad .$$

This hugeness of this number is consistent with the correspondence principle. The experiments that established the classical theory of optics weren't wrong. They were right, within their domain of applicability, in which the number of photons was so large as to be indistinguishable from a continuous beam.

Measuring the wave *example 2*

When surfers are out on the water waiting for their chance to catch a wave, they're interested in both the height of the waves and when the waves are going to arrive. In other words, they observe both the amplitude and phase of the waves, and it doesn't matter to them that the water is granular at the molecular level. The correspondence principle requires that we be able to do the same thing for electromagnetic waves, since the classical theory

Historical note

What I'm presenting in this chapter is a simplified explanation of how the photon could have been discovered. The actual history is more complex. Max Planck (1858-1947) began the photon saga with a theoretical investigation of the spectrum of light emitted by a hot, glowing object. He introduced quantization of the energy of light waves, in multiples of hf, purely as a mathematical trick that happened to produce the right results. Planck did not believe that his procedure could have any physical significance. In his 1905 paper Einstein took Planck's quantization as a description of reality, and applied it to various theoretical and experimental puzzles, including the photoelectric effect. Millikan then subjected Einstein's ideas to a series of rigorous experimental tests. Although his results matched Einstein's predictions perfectly, Millikan was skeptical about photons, and his papers conspicuously omit any reference to them. Only in his autobiography did Millikan rewrite history and claim that he had given experimental proof for photons.

of electricity and magnetism was all stated and verified experimentally in terms of the fields **E** and **B**, which are the amplitude of an electromagnetic wave. The phase is also necessary, since the laws of induction predict different results depending on whether an oscillating field is on its way up or on its way back down.

This is a more demanding application of the correspondence principle than the one in example 1, since amplitudes and phases constitute more detailed information than the over-all intensity of a beam of light. Eyeball measurements can't detect this type of information, since the eye is much bigger than a wavelength, but for example an AM radio receiver can do it with radio waves, since the wavelength for a station at 1000 kHz is about 300 meters, which is much larger than the antenna. The correspondence principle demands that we be able to explain this in terms of the photon theory, and this requires not just that we have a large number of photons emitted by the transmitter per second, as in example 1, but that even by the time they spread out and reach the receiving antenna, there should be many photons overlapping each other within a space of one cubic wavelength. Problem 13 on p. 956 verifies that the number is in fact extremely large.

Momentum of a photon *example 3*

▷ According to the theory of relativity, the momentum of a beam of light is given by $p = E/c$ (see homework problem 12 on page 787). Apply this to find the momentum of a single photon in terms of its frequency, and in terms of its wavelength.

▷ Combining the equations $p = E/c$ and $E = hf$, we find

$$\begin{aligned} p &= \frac{E}{c} \\ &= \frac{hf}{c} \quad . \end{aligned}$$

To reexpress this in terms of wavelength, we use $c = f\lambda$:

$$\begin{aligned} p &= \frac{hf}{f\lambda} \\ &= \frac{h}{\lambda} \end{aligned}$$

The second form turns out to be simpler.

Discussion questions

A The photoelectric effect only ever ejects a very tiny percentage of the electrons available near the surface of an object. How well does this agree with the wave model of light, and how well with the particle model? Consider the two different distance scales involved: the wavelength of the light, and the size of an atom, which is on the order of 10^{-10} or 10^{-9} m.

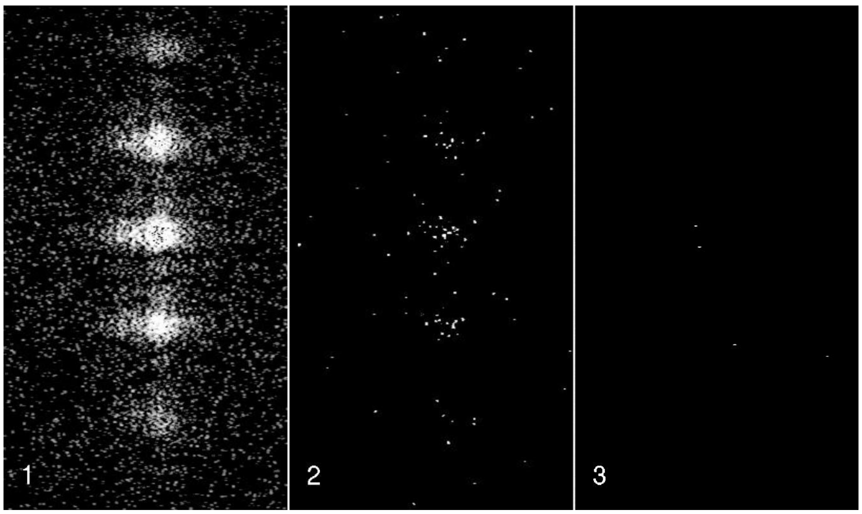
B What is the significance of the fact that Planck's constant is numerically very small? How would our everyday experience of light be different if it was not so small?

C How would the experiments described above be affected if a single electron was likely to get hit by more than one photon?

D Draw some representative trajectories of electrons for $\Delta V = 0$, ΔV less than the maximum value, and ΔV greater than the maximum value.

E Does $E = hf$ imply that a photon changes its energy when it passes from one transparent material into another substance with a different index of refraction?

34.3 Wave-particle duality

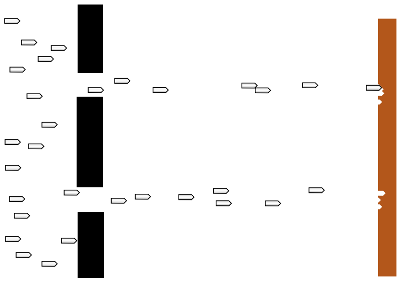


i / Wave interference patterns photographed by Prof. Lyman Page with a digital camera. Laser light with a single well-defined wavelength passed through a series of absorbers to cut down its intensity, then through a set of slits to produce interference, and finally into a digital camera chip. (A triple slit was actually used, but for conceptual simplicity we discuss the results in the main text as if it was a double slit.) In panel 2 the intensity has been reduced relative to 1, and even more so for panel 3.

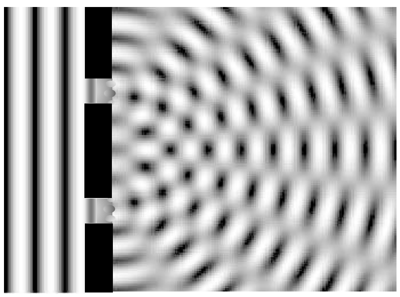
How can light be both a particle and a wave? We are now ready to resolve this seeming contradiction. Often in science when something seems paradoxical, it's because we either don't define our terms carefully, or don't test our ideas against any specific real-world situation. Let's define particles and waves as follows:

- Waves exhibit superposition, and specifically interference phenomena.
- Particles can only exist in whole numbers, not fractions.

As a real-world check on our philosophizing, there is one particular experiment that works perfectly. We set up a double-slit interference experiment that we know will produce a diffraction pattern if light is an honest-to-goodness wave, but we detect the light with a detector that is capable of sensing individual photons, e.g., a digital camera. To make it possible to pick out individual dots from individual photons, we must use filters to cut down the intensity of the light to a very low level, just as in the photos by Prof. Page in section 34.1. The whole thing is sealed inside a light-tight box. The



j / Bullets pass through a double slit.



k / A water wave passes through a double slit.

results are shown in figure i. (In fact, the similar figures in section 34.1 are simply cutouts from these figures.)

Neither the pure wave theory nor the pure particle theory can explain the results. If light was only a particle and not a wave, there would be no interference effect. The result of the experiment would be like firing a hail of bullets through a double slit, j. Only two spots directly behind the slits would be hit.

If, on the other hand, light was only a wave and not a particle, we would get the same kind of diffraction pattern that would happen with a water wave, k. There would be no discrete dots in the photo, only a diffraction pattern that shaded smoothly between light and dark.

Applying the definitions to this experiment, light must be both a particle and a wave. It is a wave because it exhibits interference effects. At the same time, the fact that the photographs contain discrete dots is a direct demonstration that light refuses to be split into units of less than a single photon. There can only be whole numbers of photons: four photons in figure i/3, for example.

A wrong interpretation: photons interfering with each other

One possible interpretation of wave-particle duality that occurred to physicists early in the game was that perhaps the interference effects came from photons interacting with each other. By analogy, a water wave consists of moving water molecules, and interference of water waves results ultimately from all the mutual pushes and pulls of the molecules. This interpretation was conclusively disproved by G.I. Taylor, a student at Cambridge. The demonstration by Prof. Page that we've just been discussing is essentially a modernized version of Taylor's work. Taylor reasoned that if interference effects came from photons interacting with each other, a bare minimum of two photons would have to be present at the same time to produce interference. By making the light source extremely dim, we can be virtually certain that there are never two photons in the box at the same time. In figure i, the intensity of the light has been cut down so much by the absorbers that if it was in the open, the average separation between photons would be on the order of a kilometer! At any given moment, the number of photons in the box is most likely to be zero. It is virtually certain that there were never two photons in the box at once.

The concept of a photon's path is undefined.

If a single photon can demonstrate double-slit interference, then which slit did it pass through? The unavoidable answer must be that it passes through both! This might not seem so strange if we think of the photon as a wave, but it is highly counterintuitive if we try to visualize it as a particle. The moral is that we should not think in terms of the *path* of a photon. Like the fully human and fully divine Jesus of Christian theology, a photon is supposed to be 100% wave and 100% particle. If a photon had a well defined path, then it would not demonstrate wave superposition and interference effects, contradicting its wave nature. (In the next chapter we will discuss the Heisenberg uncertainty principle, which gives a numerical way of approaching this issue.)

l / A single photon can go through both slits.

Another wrong interpretation: the pilot wave hypothesis

A second possible explanation of wave-particle duality was taken seriously in the early history of quantum mechanics. What if the photon *particle* is like a surfer riding on top of its accompanying *wave*? As the wave travels along, the particle is pushed, or "piloted" by it. Imagining the particle and the wave as two separate entities allows us to avoid the seemingly paradoxical idea that a photon is both at once. The wave happily does its wave tricks, like superposition and interference, and the particle acts like a respectable particle, resolutely refusing to be in two different places at once. If the wave, for instance, undergoes destructive interference, becoming nearly zero in a particular region of space, then the particle simply is not guided into that region.

The problem with the pilot wave interpretation is that the only way it can be experimentally tested or verified is if someone manages to detach the particle from the wave, and show that there really are two entities involved, not just one. Part of the scientific method is that hypotheses are supposed to be experimentally testable. Since nobody has ever managed to separate the wavelike part of a photon from the particle part, the interpretation is not useful or meaningful in a scientific sense.

The probability interpretation

The correct interpretation of wave-particle duality is suggested by the random nature of the experiment we've been discussing: even though every photon wave/particle is prepared and released in the same way, the location at which it is eventually detected by the digital camera is different every time. The idea of the probability interpretation of wave-particle duality is that the location of the photon-particle is random, but the probability that it is in a certain location is higher where the photon-wave's amplitude is greater.

More specifically, the probability distribution of the particle must be proportional to the *square* of the wave's amplitude,

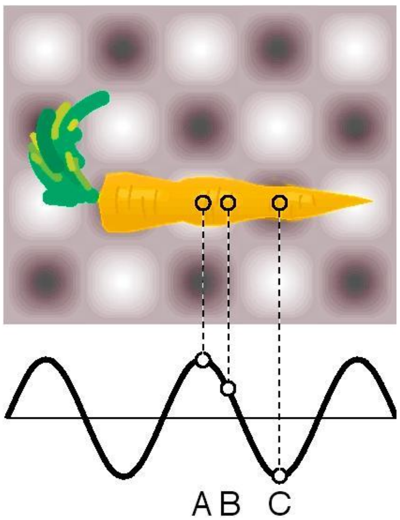
$$\text{(probability distribution)} \propto \text{(amplitude)}^2 \qquad .$$

This follows from the correspondence principle and from the fact that a wave's energy density is proportional to the square of its amplitude. If we run the double-slit experiment for a long enough time, the pattern of dots fills in and becomes very smooth as would have been expected in classical physics. To preserve the correspondence between classical and quantum physics, the amount of energy deposited in a given region of the picture over the long run must be proportional to the square of the wave's amplitude. The amount of energy deposited in a certain area depends on the number of photons picked up, which is proportional to the probability of finding any given photon there.

A microwave oven *example 4*

▷ The figure shows two-dimensional (top) and one-dimensional (bottom) representations of the standing wave inside a microwave oven. Gray represents zero field, and white and black signify the strongest fields, with white being a field that is in the opposite direction compared to black. Compare the probabilities of detecting a microwave photon at points A, B, and C.

▷ A and C are both extremes of the wave, so the probabilities of detecting a photon at A and C are equal. It doesn't matter that we have represented C as negative and A as positive, because it is the square of the amplitude that is relevant. The amplitude at B is about 1/2 as much as the others, so the probability of detecting a photon there is about 1/4 as much.



m / Example 4.

The probability interpretation was disturbing to physicists who had spent their previous careers working in the deterministic world of classical physics, and ironically the most strenuous objections against it were raised by Einstein, who had invented the photon concept in the first place. The probability interpretation has nevertheless passed every experimental test, and is now as well established as any part of physics.

An aspect of the probability interpretation that has made many people uneasy is that the process of detecting and recording the photon's position seems to have a magical ability to get rid of the wavelike side of the photon's personality and force it to decide for once and for all where it really wants to be. But detection or measurement is after all only a physical process like any other, governed by the same laws of physics. We will postpone a detailed discussion of this issue until the following chapter, since a measuring device like a digital camera is made of matter, but we have so far only discussed how quantum mechanics relates to light.

What is the proportionality constant? *example 5*

▷ What is the proportionality constant that would make an actual equation out of (probability distribution) $\propto$ (amplitude)^2?

▷ The probability that the photon is in a certain small region of volume v should equal the fraction of the wave's energy that is within that volume. For a sinusoidal wave, which has a single, well-defined frequency f, this gives

$$P = \frac{\text{energy in volume } v}{\text{energy of photon}} = \frac{\text{energy in volume } v}{hf} \quad .$$

We assume v is small enough so that the electric and magnetic fields are nearly constant throughout it. We then have

$$P = \frac{\left(\frac{1}{8\pi k}|\mathbf{E}|^2 + \frac{c^2}{8\pi k}|\mathbf{B}|^2\right) v}{hf} \quad .$$

We can simplify this formidable looking expression by recognizing that in a plane wave, $|\mathbf{E}|$ and $|\mathbf{B}|$ are related by $|\mathbf{E}| = c|\mathbf{B}|$. This implies (problem 20, p. 703), that the electric and magnetic fields each contribute half the total energy, so we can simplify the result to

$$P = 2\frac{\left(\frac{1}{8\pi k}|\mathbf{E}|^2\right) v}{hf} = \frac{v}{4\pi khf}|\mathbf{E}|^2 \quad .$$

The probability is proportional to the square of the wave's amplitude, as advertised.[3]

Discussion questions

A In example 4 on page 950, about the carrot in the microwave oven, show that it would be nonsensical to have probability be proportional to the field itself, rather than the square of the field.

B Einstein did not try to reconcile the wave and particle theories of light, and did not say much about their apparent inconsistency. Einstein basically visualized a beam of light as a stream of bullets coming from a machine gun. In the photoelectric effect, a photon "bullet" would only hit one atom, just as a real bullet would only hit one person. Suppose someone reading his 1905 paper wanted to interpret it by saying that Einstein's so-called particles of light are simply short wave-trains that only occupy a small region of space. Comparing the wavelength of visible light

[3]But note that along the way, we had to make two crucial assumptions: that the wave was sinusoidal, and that it was a plane wave. These assumptions will not prevent us from describing examples such as double-slit diffraction, in which the wave is approximately sinusoidal within some sufficiently small region such as one pixel of a camera's imaging chip. Nevertheless, these issues turn out to be symptoms of deeper problems, beyond the scope of this book, involving the way in which relativity and quantum mechanics should be combined. As a taste of the ideas involved, consider what happens when a photon is reflected from a conducting surface, as in example 8 on p. 688, so that the electric field at the surface is zero, but the magnetic field isn't. The superposition is a standing wave, not a plane wave, so $|\mathbf{E}| = c|\mathbf{B}|$ need not hold, and doesn't. A detector's probability of detecting a photon near the surface could be zero if the detector sensed electric fields, but nonzero if it sensed magnetism. It doesn't make sense to say that either of these is the probability that the photon "was really there."

(a few hundred nm) to the size of an atom (on the order of 0.1 nm), explain why this poses a difficulty for reconciling the particle and wave theories.

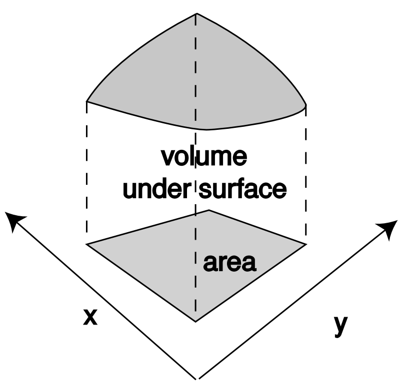
C Can a white photon exist?

D In double-slit diffraction of photons, would you get the same pattern of dots on the digital camera image if you covered one slit? Why should it matter whether you give the photon two choices or only one?

34.4 Photons in three dimensions

Up until now I've been sneaky and avoided a full discussion of the three-dimensional aspects of the probability interpretation. The example of the carrot in the microwave oven, for example, reduced to a one-dimensional situation because we were considering three points along the same line and because we were only comparing ratios of probabilities. The purpose of bringing it up now is to head off any feeling that you've been cheated conceptually rather than to prepare you for mathematical problem solving in three dimensions, which would not be appropriate for the level of this course.

A typical example of a probability distribution in section 33.3 was the distribution of heights of human beings. The thing that varied randomly, height, h, had units of meters, and the probability distribution was a graph of a function $D(h)$. The units of the probability distribution had to be m^{-1} (inverse meters) so that areas under the curve, interpreted as probabilities, would be unitless: (area) = (height)(width) = $\text{m}^{-1} \cdot \text{m}$.



n / The volume under a surface.

Now suppose we have a two-dimensional problem, e.g., the probability distribution for the place on the surface of a digital camera chip where a photon will be detected. The point where it is detected would be described with two variables, x and y, each having units of meters. The probability distribution will be a function of both variables, $D(x, y)$. A probability is now visualized as the volume under the surface described by the function $D(x, y)$, as shown in figure n. The units of D must be m^{-2} so that probabilities will be unitless: (probability) = (depth)(length)(width) = $\text{m}^{-2} \cdot \text{m} \cdot \text{m}$.

Generalizing finally to three dimensions, we find by analogy that the probability distribution will be a function of all three coordinates, $D(x, y, z)$, and will have units of m^{-3}. It is, unfortunately, impossible to visualize the graph unless you are a mutant with a natural feel for life in four dimensions. If the probability distribution is nearly constant within a certain volume of space v, the probability that the photon is in that volume is simply vD. If you know enough calculus, it should be clear that this can be generalized to $P = \int D \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$ if D is not constant.

Summary

Selected vocabulary

photon	a particle of light
photoelectric effect	the ejection, by a photon, of an electron from the surface of an object
wave-particle duality . . .	the idea that light is both a wave and a particle

Summary

Around the turn of the twentieth century, experiments began to show problems with the classical wave theory of light. In any experiment sensitive enough to detect very small amounts of light energy, it becomes clear that light energy cannot be divided into chunks smaller than a certain amount. Measurements involving the photoelectric effect demonstrate that this smallest unit of light energy equals hf, where f is the frequency of the light and h is a number known as Planck's constant. We say that light energy is quantized in units of hf, and we interpret this quantization as evidence that light has particle properties as well as wave properties. Particles of light are called photons.

The only method of reconciling the wave and particle natures of light that has stood the test of experiment is the probability interpretation: the probability that the particle is at a given location is proportional to the square of the amplitude of the wave at that location.

One important consequence of wave-particle duality is that we must abandon the concept of the path the particle takes through space. To hold on to this concept, we would have to contradict the well established wave nature of light, since a wave can spread out in every direction simultaneously.

Problems

Key

√ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

For some of these homework problems, you may find it convenient to refer to the diagram of the electromagnetic spectrum shown on p. 689.

1 Give a numerical comparison of the number of photons per second emitted by a hundred-watt FM radio transmitter and a hundred-watt lightbulb. √

2 Two different flashes of light each have the same energy. One consists of photons with a wavelength of 600 nm, the other 400 nm. If the number of photons in the 600-nm flash is 3.0×10^{18}, how many photons are in the 400-nm flash? √

3 When light is reflected from a mirror, perhaps only 80% of the energy comes back. The rest is converted to heat. One could try to explain this in two different ways: (1) 80% of the photons are reflected, or (2) all the photons are reflected, but each loses 20% of its energy. Based on your everyday knowledge about mirrors, how can you tell which interpretation is correct? [Based on a problem from PSSC Physics.]

4 Suppose we want to build an electronic light sensor using an apparatus like the one described in section 34.2 on p. 942. How would its ability to detect different parts of the spectrum depend on the type of metal used in the capacitor plates?

5 The photoelectric effect can occur not just for metal cathodes but for any substance, including living tissue. Ionization of DNA molecules can cause cancer or birth defects. If the energy required to ionize DNA is on the same order of magnitude as the energy required to produce the photoelectric effect in a metal, which of these types of electromagnetic waves might pose such a hazard? Explain.

60 Hz waves from power lines

100 MHz FM radio

1900 MHz radio waves from a cellular phone

2450 MHz microwaves from a microwave oven

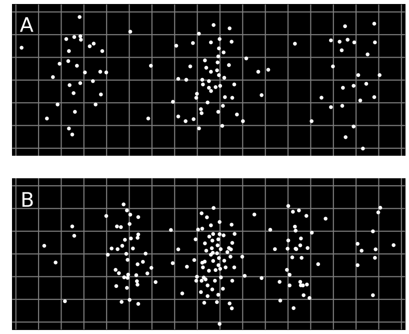
visible light

ultraviolet light

x-rays

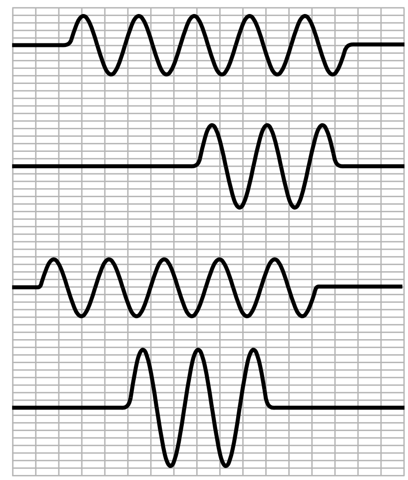
6 The beam of a 100-W overhead projector covers an area of 1 m × 1 m when it hits the screen 3 m away. Estimate the number of photons that are in flight at any given time. (Since this is only an estimate, we can ignore the fact that the beam is not parallel.)

7 The two diffraction patterns were made by sending a flash of light through the same double slit. Give a numerical comparison of the amounts of energy in the two flashes. √



Problem 7.

8 Three of the four graphs are properly normalized to represent single photons. Which one isn't? Explain.



Problem 8.

9 Photon Fred has a greater energy than photon Ginger. For each of the following quantities, explain whether Fred's value of that quantity is greater than Ginger's, less than Ginger's, or equal to Ginger's. If there is no way to tell, explain why.

frequency

speed

wavelength

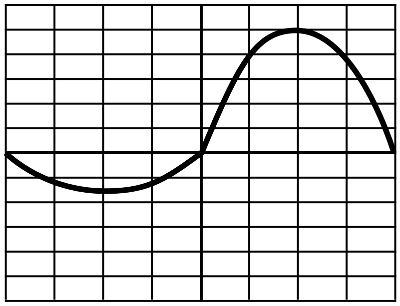
period

electric field strength

magnetic field strength

10 Give experimental evidence to disprove the following interpretation of wave-particle duality: *A photon is really a particle, but it travels along a wavy path, like a zigzag with rounded corners.* Cite a specific, real experiment.

11 In the photoelectric effect, electrons are observed with virtually no time delay ($\sim$ 10 ns), even when the light source is very weak. (A weak light source does however only produce a small number of ejected electrons.) The purpose of this problem is to show that the lack of a significant time delay contradicted the classical wave theory of light, so throughout this problem you should put yourself in the shoes of a classical physicist and pretend you don't know about photons at all. At that time, it was thought that the electron might have a radius on the order of 10^{-15} m. (Recent experiments have shown that if the electron has any finite size at all, it is far smaller.)
(a) Estimate the power that would be soaked up by a single electron in a beam of light with an intensity of 1 mW/m^2. √
(b) The energy, E_s, required for the electron to escape through the surface of the cathode is on the order of 10^{-19} J. Find how long it would take the electron to absorb this amount of energy, and explain why your result constitutes strong evidence that there is something wrong with the classical theory. √



Problem 12.

12 Many radio antennas are designed so that they preferentially emit or receive electromagnetic waves in a certain direction. However, no antenna is perfectly directional. The wave shown in the figure represents a single photon being emitted by an antenna at the center. The antenna is directional, so there is a stronger wave on the right than on the left. What is the probability that the photon will be observed on the right?

13 (a) A radio transmitter radiates power P in all directions, so that the energy spreads out spherically. Find the energy density at a distance r. ✓
(b) Let the wavelength be λ. As described in example 2 on p. 945, find the number of photons in a volume λ^3 at this distance r. ✓
(c) For a 1000 kHz AM radio transmitting station, assuming reasonable values of P and r, verify, as claimed in the example, that the result from part b is very large.

Dorothy melts the Wicked Witch of the West.

Chapter 35
Matter as a wave

> [In] a few minutes I shall be all melted... I have been wicked in my day, but I never thought a little girl like you would ever be able to melt me and end my wicked deeds. Look out — here I go!
>
> *The Wicked Witch of the West*

As the Wicked Witch learned the hard way, losing molecular cohesion can be unpleasant. That's why we should be very grateful that the concepts of quantum physics apply to matter as well as light. If matter obeyed the laws of classical physics, molecules wouldn't exist.

Consider, for example, the simplest atom, hydrogen. Why does one hydrogen atom form a chemical bond with another hydrogen atom? Roughly speaking, we'd expect a neighboring pair of hydrogen atoms, A and B, to exert no force on each other at all, attractive or repulsive: there are two repulsive interactions (proton A with proton B and electron A with electron B) and two attractive interactions (proton A with electron B and electron A with proton B). Thinking a little more precisely, we should even expect that once the two atoms got close enough, the interaction would be repulsive. For instance, if you squeezed them so close together that the two protons were almost on top of each other, there would be a tremendously strong repulsion between them due to the $1/r^2$ nature of the electrical force. The repulsion between the electrons would not be as strong, because each electron ranges over a large area, and is not likely to be found right on top of the other electron. This was only a rough argument based on averages, but the conclusion is validated by a more complete classical analysis: hydrogen molecules should not exist according to classical physics.

Quantum physics to the rescue! As we'll see shortly, the whole problem is solved by applying the same quantum concepts to electrons that we have already used for photons.

35.1 Electrons as waves

We started our journey into quantum physics by studying the random behavior of *matter* in radioactive decay, and then asked how randomness could be linked to the basic laws of nature governing *light*. The probability interpretation of wave-particle duality was strange and hard to accept, but it provided such a link. It is now natural to ask whether the same explanation could be applied to matter. If the fundamental building block of light, the photon, is a particle as well as a wave, is it possible that the basic units of matter, such as electrons, are waves as well as particles?

A young French aristocrat studying physics, Louis de Broglie (pronounced "broylee"), made exactly this suggestion in his 1923 Ph.D. thesis. His idea had seemed so farfetched that there was serious doubt about whether to grant him the degree. Einstein was asked for his opinion, and with his strong support, de Broglie got his degree.

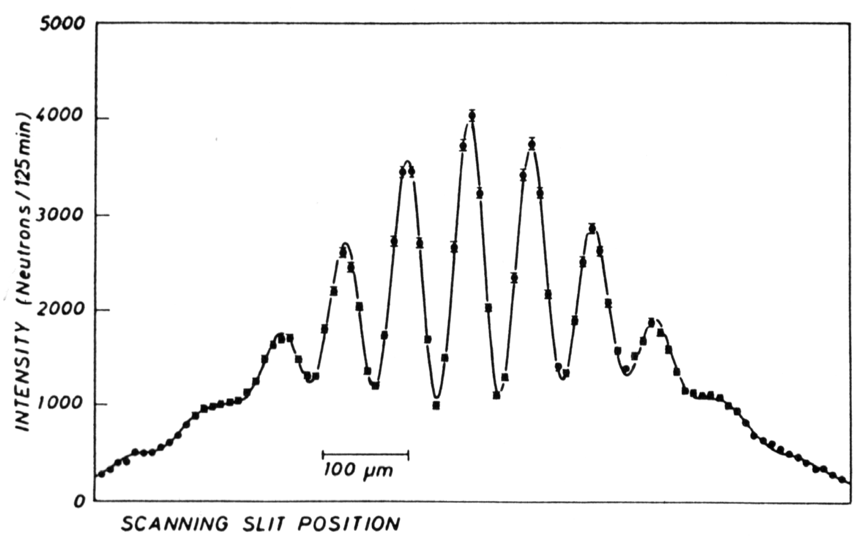
Only two years later, American physicists C.J. Davisson and L. Germer confirmed de Broglie's idea by accident. They had been studying the scattering of electrons from the surface of a sample of nickel, made of many small crystals. (One can often see such a crystalline pattern on a brass doorknob that has been polished by repeated handling.) An accidental explosion occurred, and when they put their apparatus back together they observed something entirely different: the scattered electrons were now creating an interference pattern! This dramatic proof of the wave nature of matter came about because the nickel sample had been melted by the explosion and then resolidified as a single crystal. The nickel atoms, now nicely arranged in the regular rows and columns of a crystalline lattice, were acting as the lines of a diffraction grating. The new crystal was analogous to the type of ordinary diffraction grating in which the lines are etched on the surface of a mirror (a reflection grating) rather than the kind in which the light passes through the transparent gaps between the lines (a transmission grating).

Although we will concentrate on the wave-particle duality of electrons because it is important in chemistry and the physics of atoms, all the other "particles" of matter you've learned about show wave properties as well. Figure a, for instance, shows a wave interference pattern of neutrons.

It might seem as though all our work was already done for us, and there would be nothing new to understand about electrons: they have the same kind of funny wave-particle duality as photons. That's almost true, but not quite. There are some important ways in which electrons differ significantly from photons:

1. Electrons have mass, and photons don't.
2. Photons always move at the speed of light, but electrons can move at any speed less than c.
3. Photons don't have electric charge, but electrons do, so electric forces can act on them. The most important example is the atom, in which the electrons are held by the electric force of the nucleus.
4. Electrons cannot be absorbed or emitted as photons are. Destroying an electron, or creating one out of nothing, would violate conservation of charge.

(In chapter 36 we will learn of one more fundamental way in which electrons differ from photons, for a total of five.)



a / A double-slit interference pattern made with neutrons. (A. Zeilinger, R. Gähler, C.G. Shull, W. Treimer, and W. Mampe, Reviews of Modern Physics, Vol. 60, 1988.)

Because electrons are different from photons, it is not immediately obvious which of the photon equations from chapter 34 can be applied to electrons as well. A particle property, the energy of one photon, is related to its wave properties via $E = hf$ or, equivalently, $E = hc/\lambda$. The momentum of a photon was given by $p = hf/c$ or

$p = h/\lambda$ (example 3 on page 946). Ultimately it was a matter of experiment to determine which of these equations, if any, would work for electrons, but we can make a quick and dirty guess simply by noting that some of the equations involve c, the speed of light, and some do not. Since c is irrelevant in the case of an electron, we might guess that the equations of general validity are those that do not have c in them:

$$E = hf$$
$$p = \frac{h}{\lambda}$$

This is essentially the reasoning that de Broglie went through, and experiments have confirmed these two equations for all the fundamental building blocks of light and matter, not just for photons and electrons.

The second equation, which I soft-pedaled in chapter 34, takes on a greater importance for electrons. This is first of all because the momentum of matter is more likely to be significant than the momentum of light under ordinary conditions, and also because force is the transfer of momentum, and electrons are affected by electrical forces.

The wavelength of an elephant *example 1*

▷ What is the wavelength of a trotting elephant?

▷ One may doubt whether the equation should be applied to an elephant, which is not just a single particle but a rather large collection of them. Throwing caution to the wind, however, we estimate the elephant's mass at 10^3 kg and its trotting speed at 10 m/s. Its wavelength is therefore roughly

$$\begin{aligned}\lambda &= \frac{h}{p}\\ &= \frac{h}{mv}\\ &= \frac{6.63\times10^{-34}\ \text{J}\cdot\text{s}}{(10^3\ \text{kg})(10\ \text{m/s})}\\ &\sim 10^{-37}\ \frac{(\text{kg}\cdot\text{m}^2/\text{s}^2)\cdot\text{s}}{\text{kg}\cdot\text{m/s}}\\ &= 10^{-37}\ \text{m} \quad .\end{aligned}$$

The wavelength found in this example is so fantastically small that we can be sure we will never observe any measurable wave phenomena with elephants. The result is numerically small because Planck's constant is so small, and as in some examples encountered previously, this smallness is in accord with the correspondence principle.

Although a smaller mass in the equation $\lambda = h/mv$ does result in a longer wavelength, the wavelength is still quite short even for individual electrons under typical conditions, as shown in the following example.

The typical wavelength of an electron *example 2*

▷ Electrons in circuits and in atoms are typically moving through voltage differences on the order of 1 V, so that a typical energy is $(e)(1\text{ V})$, which is on the order of 10^{-19} J. What is the wavelength of an electron with this amount of kinetic energy?

▷ This energy is nonrelativistic, since it is much less than mc^2. Momentum and energy are therefore related by the nonrelativistic equation $KE = p^2/2m$. Solving for p and substituting in to the equation for the wavelength, we find

$$\begin{aligned}\lambda &= \frac{h}{\sqrt{2m \cdot KE}} \\ &= 1.6 \times 10^{-9}\text{ m} \quad .\end{aligned}$$

This is on the same order of magnitude as the size of an atom, which is no accident: as we will discuss in the next chapter in more detail, an electron in an atom can be interpreted as a standing wave. The smallness of the wavelength of a typical electron also helps to explain why the wave nature of electrons wasn't discovered until a hundred years after the wave nature of light. To scale the usual wave-optics devices such as diffraction gratings down to the size needed to work with electrons at ordinary energies, we need to make them so small that their parts are comparable in size to individual atoms. This is essentially what Davisson and Germer did with their nickel crystal.

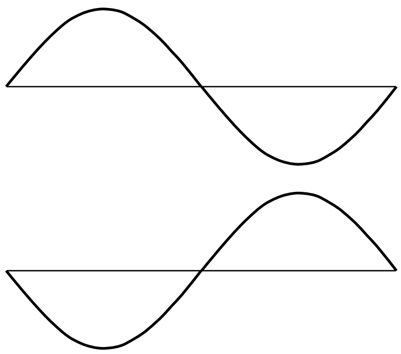
self-check A

These remarks about the inconvenient smallness of electron wavelengths apply only under the assumption that the electrons have typical energies. What kind of energy would an electron have to have in order to have a longer wavelength that might be more convenient to work with?
▷ Answer, p. 1011

What kind of wave is it?

If a sound wave is a vibration of matter, and a photon is a vibration of electric and magnetic fields, what kind of a wave is an electron made of? The disconcerting answer is that there is no experimental "observable," i.e., directly measurable quantity, to correspond to the electron wave itself. In other words, there are devices like microphones that detect the oscillations of air pressure in a sound wave, and devices such as radio receivers that measure the oscillation of the electric and magnetic fields in a light wave, but nobody has ever found any way to measure an electron wave directly.

We can of course detect the energy (or momentum) possessed by an electron just as we could detect the energy of a photon using a digital camera. (In fact I'd imagine that an unmodified digital camera chip placed in a vacuum chamber would detect electrons just as handily as photons.) But this only allows us to determine where the wave carries high probability and where it carries low probability. Probability is proportional to the square of the wave's amplitude, but measuring its square is not the same as measuring the wave itself. In particular, we get the same result by squaring either a positive number or its negative, so there is no way to determine the positive or negative sign of an electron wave.



b / These two electron waves are not distinguishable by any measuring device.

Most physicists tend toward the school of philosophy known as operationalism, which says that a concept is only meaningful if we can define some set of operations for observing, measuring, or testing it. According to a strict operationalist, then, the electron wave itself is a meaningless concept. Nevertheless, it turns out to be one of those concepts like love or humor that is impossible to measure and yet very useful to have around. We therefore give it a symbol, Ψ (the capital Greek letter psi), and a special name, the electron *wavefunction* (because it is a function of the coordinates x, y, and z that specify where you are in space). It would be impossible, for example, to calculate the shape of the electron wave in a hydrogen atom without having some symbol for the wave. But when the calculation produces a result that can be compared directly to experiment, the final algebraic result will turn out to involve only Ψ^2, which is what is observable, not Ψ itself.

Since Ψ, unlike $\mathbf{E}$ and $\mathbf{B}$, is not directly measurable, we are free to make the probability equations have a simple form: instead of having the probability distribution equal to some funny constant multiplied by Ψ^2, we simply define Ψ so that the constant of proportionality is one:

$$(\text{probability distribution}) = \Psi^2 \qquad .$$

Since the probability distribution has units of m^{-3}, the units of Ψ must be $\mathrm{m}^{-3/2}$.

Discussion question

A Frequency is oscillations per second, whereas wavelength is meters per oscillation. How could the equations $E = hf$ and $p = h/\lambda$ be made to look more alike by using quantities that were more closely analogous? (This more symmetric treatment makes it easier to incorporate relativity into quantum mechanics, since relativity says that space and time are not entirely separate.)

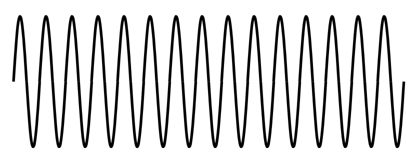
35.2 $\int\star$ Dispersive waves

A colleague of mine who teaches chemistry loves to tell the story about an exceptionally bright student who, when told of the equation $p = h/\lambda$, protested, "But when I derived it, it had a factor of 2!" The issue that's involved is a real one, albeit one that could be glossed over (and is, in most textbooks) without raising any alarms in the mind of the average student. The present optional section addresses this point; it is intended for the student who wishes to delve a little deeper.

Here's how the now-legendary student was presumably reasoning. We start with the equation $v = f\lambda$, which is valid for any sine wave, whether it's quantum or classical. Let's assume we already know $E = hf$, and are trying to derive the relationship between wavelength and momentum:

$$\begin{aligned}\lambda &= \frac{v}{f}\\ &= \frac{vh}{E}\\ &= \frac{vh}{\frac{1}{2}mv^2}\\ &= \frac{2h}{mv}\\ &= \frac{2h}{p}\end{aligned}$$

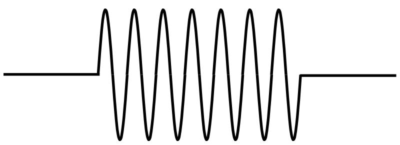
The reasoning seems valid, but the result does contradict the accepted one, which is after all solidly based on experiment.



c / Part of an infinite sine wave.

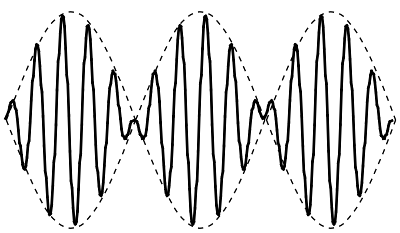
The mistaken assumption is that we can figure everything out in terms of pure sine waves. Mathematically, the only wave that has a perfectly well defined wavelength and frequency is a sine wave, and not just any sine wave but an infinitely long one, c. The unphysical thing about such a wave is that it has no leading or trailing edge, so it can never be said to enter or leave any particular region of space. Our derivation made use of the velocity, v, and if velocity is to be a meaningful concept, it must tell us how quickly stuff (mass, energy, momentum,...) is transported from one region of space to another. Since an infinitely long sine wave doesn't remove any stuff from one region and take it to another, the "velocity of its stuff" is not a well defined concept.

Of course the individual wave peaks do travel through space, and one might think that it would make sense to associate their speed with the "speed of stuff," but as we will see, the two velocities are in general unequal when a wave's velocity depends on wavelength. Such a wave is called a *dispersive* wave, because a wave pulse consisting of a superposition of waves of different wavelengths will separate (disperse) into its separate wavelengths as the waves move through space at different speeds. Nearly all the waves we have encountered have been nondispersive. For instance, sound waves and light waves (in a vacuum) have speeds independent of wavelength. A water wave is one good example of a dispersive wave. Long-wavelength water waves travel faster, so a ship at sea that encounters a storm typically sees the long-wavelength parts of the wave first. When dealing with dispersive waves, we need symbols and words to distinguish the two speeds. The speed at which wave peaks move is called the phase velocity, v_p, and the speed at which "stuff" moves is called the group velocity, v_g.



d / A finite-length sine wave.

An infinite sine wave can only tell us about the phase velocity, not the group velocity, which is really what we would be talking about when we referred to the speed of an electron. If an infinite sine wave is the simplest possible wave, what's the next best thing? We might think the runner up in simplicity would be a wave train consisting of a chopped-off segment of a sine wave, d. However, this kind of wave has kinks in it at the end. A simple wave should be one that we can build by superposing a small number of infinite sine waves, but a kink can never be produced by superposing any number of infinitely long sine waves.



e / A beat pattern created by superimposing two sine waves with slightly different wavelengths.

Actually the simplest wave that transports stuff from place to place is the pattern shown in figure e. Called a beat pattern, it is formed by superposing two sine waves whose wavelengths are similar but not quite the same. If you have ever heard the pulsating howling sound of musicians in the process of tuning their instruments to each other, you have heard a beat pattern. The beat pattern gets stronger and weaker as the two sine waves go in and out of phase with each other. The beat pattern has more "stuff" (energy, for example) in the areas where constructive interference occurs, and less in the regions of cancellation. As the whole pattern moves through space, stuff is transported from some regions and into other ones.

If the frequency of the two sine waves differs by 10%, for instance, then ten periods will be occur between times when they are in phase. Another way of saying it is that the sinusoidal "envelope" (the dashed lines in figure e) has a frequency equal to the difference in frequency between the two waves. For instance, if the waves had frequencies of 100 Hz and 110 Hz, the frequency of the envelope would be 10 Hz.

To apply similar reasoning to the wavelength, we must define a quantity $z = 1/\lambda$ that relates to wavelength in the same way that frequency relates to period. In terms of this new variable, the z of the envelope equals the difference between the z's of the two sine waves.

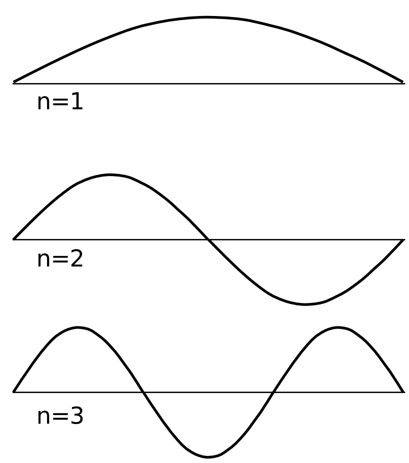
The group velocity is the speed at which the envelope moves through space. Let Δf and Δz be the differences between the frequencies and z's of the two sine waves, which means that they equal the frequency and z of the envelope. The group velocity is $v_g = f_{envelope}\lambda_{envelope} = \Delta f/\Delta\mathrm{z}$. If Δf and Δz are sufficiently small, we can approximate this expression as a derivative,

$$v_g = \frac{\mathrm{d}f}{\mathrm{d}z} \qquad .$$

This expression is usually taken as the definition of the group velocity for wave patterns that consist of a superposition of sine waves having a narrow range of frequencies and wavelengths. In quantum mechanics, with $f = E/h$ and $z = p/h$, we have $v_g = \mathrm{d}E/\,\mathrm{d}p$. In the case of a nonrelativistic electron the relationship between energy and momentum is $E = p^2/2m$, so the group velocity is $\mathrm{d}E/\,\mathrm{d}p = p/m = v$, exactly what it should be. It is only the phase velocity that differs by a factor of two from what we would have expected, but the phase velocity is not the physically important thing.

35.3 Bound states

Electrons are at their most interesting when they're in atoms, that is, when they are bound within a small region of space. We can understand a great deal about atoms and molecules based on simple arguments about such bound states, without going into any of the realistic details of atom. The simplest model of a bound state is known as the particle in a box: like a ball on a pool table, the electron feels zero force while in the interior, but when it reaches an edge it encounters a wall that pushes back inward on it with a large force. In particle language, we would describe the electron as bouncing off of the wall, but this incorrectly assumes that the electron has a certain path through space. It is more correct to describe the electron as a wave that undergoes 100% reflection at the boundaries of the box.



f / Three possible standing-wave patterns for a particle in a box.

Like a generation of physics students before me, I rolled my eyes when initially introduced to the unrealistic idea of putting a particle in a box. It seemed completely impractical, an artificial textbook invention. Today, however, it has become routine to study electrons in rectangular boxes in actual laboratory experiments. The "box" is actually just an empty cavity within a solid piece of silicon, amounting in volume to a few hundred atoms. The methods for creating these electron-in-a-box setups (known as "quantum dots") were a by-product of the development of technologies for fabricating computer chips.

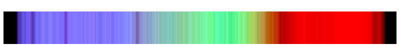
For simplicity let's imagine a one-dimensional electron in a box, i.e., we assume that the electron is only free to move along a line. The resulting standing wave patterns, of which the first three are shown in figure f, are just like some of the patterns we encountered with sound waves in musical instruments. The wave patterns must be zero at the ends of the box, because we are assuming the walls are impenetrable, and there should therefore be zero probability of finding the electron outside the box. Each wave pattern is labeled according to n, the number of peaks and valleys it has. In quantum physics, these wave patterns are referred to as "states" of the particle-in-the-box system.

The following seemingly innocuous observations about the particle in the box lead us directly to the solutions to some of the most vexing failures of classical physics:

The particle's energy is quantized (can only have certain values). Each wavelength corresponds to a certain momentum, and a given momentum implies a definite kinetic energy, $E = p^2/2m$. (This is the second type of energy quantization we have encountered. The type we studied previously had to do with restricting the number of particles to a whole number, while assuming some specific wavelength and energy for each particle. This type of quantization refers to the energies that a single particle can have. Both photons and matter particles demonstrate both types of quantization under the appropriate circumstances.)

The particle has a minimum kinetic energy. Long wavelengths correspond to low momenta and low energies. There can be no state with an energy lower than that of the $n = 1$ state, called the ground state.

The smaller the space in which the particle is confined, the higher its kinetic energy must be. Again, this is because long wavelengths give lower energies.



g / The spectrum of the light from the star Sirius. Photograph by the author.

Spectra of thin gases — *example 3*

A fact that was inexplicable by classical physics was that thin gases absorb and emit light only at certain wavelengths. This was observed both in earthbound laboratories and in the spectra of stars. Figure g shows the example of the spectrum of the star Sirius, in which there are "gap teeth" at certain wavelengths. Taking this spectrum as an example, we can give a straightforward explanation using quantum physics.

Energy is released in the dense interior of the star, but the outer layers of the star are thin, so the atoms are far apart and electrons are confined within individual atoms. Although their standing-wave patterns are not as simple as those of the particle in the box, their energies are quantized.

When a photon is on its way out through the outer layers, it can be absorbed by an electron in an atom, but only if the amount of energy it carries happens to be the right amount to kick the electron from one of the allowed energy levels to one of the higher levels. The photon energies that are missing from the spectrum are the ones that equal the difference in energy between two electron energy levels. (The most prominent of the absorption lines in

Sirius's spectrum are absorption lines of the hydrogen atom.)

The stability of atoms *example 4*

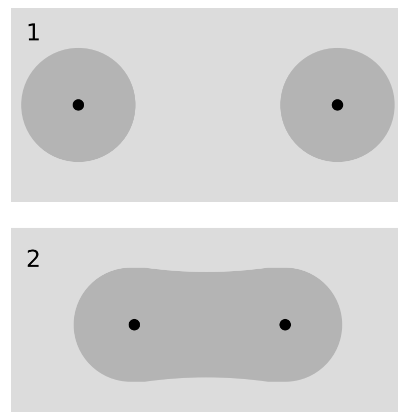
In many Star Trek episodes the Enterprise, in orbit around a planet, suddenly lost engine power and began spiraling down toward the planet's surface. This was utter nonsense, of course, due to conservation of energy: the ship had no way of getting rid of energy, so it did not need the engines to replenish it.

Consider, however, the electron in an atom as it orbits the nucleus. The electron *does* have a way to release energy: it has an acceleration due to its continuously changing direction of motion, and according to classical physics, any accelerating charged particle emits electromagnetic waves. According to classical physics, atoms should collapse!

The solution lies in the observation that a bound state has a minimum energy. An electron in one of the higher-energy atomic states can and does emit photons and hop down step by step in energy. But once it is in the ground state, it cannot emit a photon because there is no lower-energy state for it to go to.

Chemical bonds in hydrogen molecules *example 5*

I began this chapter with a classical argument that chemical bonds, as in an H_2 molecule, should not exist. Quantum physics explains why this type of bonding does in fact occur. When the atoms are next to each other, the electrons are shared between them. The "box" is about twice as wide, and a larger box allows a smaller energy. Energy is required in order to separate the atoms. (A qualitatively different type of bonding is discussed in on page 995.)



h / Example 5: Two hydrogen atoms bond to form an H_2 molecule. In the molecule, the two electrons' wave patterns overlap, and are about twice as wide.

Discussion questions

A Neutrons attract each other via the strong nuclear force, so according to classical physics it should be possible to form nuclei out of clusters of two or more neutrons, with no protons at all. Experimental searches, however, have failed to turn up evidence of a stable two-neutron system (dineutron) or larger stable clusters. These systems are apparently not just unstable in the sense of being able to beta decay but unstable in the sense that they don't hold together at all. Explain based on quantum physics why a dineutron might spontaneously fly apart.

B The following table shows the energy gap between the ground state and the first excited state for four nuclei, in units of picojoules. (The nuclei were chosen to be ones that have similar structures, e.g., they are all spherical in shape.)

nucleus	energy gap (picojoules)
^{4}He	3.234
^{16}O	0.968
^{40}Ca	0.536
^{208}Pb	0.418

Explain the trend in the data.

35.4 The uncertainty principle

The uncertainty principle

Eliminating randomness through measurement?

i / Werner Heisenberg (1901-1976). Heisenberg helped to develop the foundations of quantum mechanics, including the Heisenberg uncertainty principle. He was the scientific leader of the Nazi atomic-bomb program up until its cancellation in 1942, when the military decided that it was too ambitious a project to undertake in wartime, and too unlikely to produce results.

A common reaction to quantum physics, among both early-twentieth-century physicists and modern students, is that we should be able to get rid of randomness through accurate measurement. If I say, for example, that it is meaningless to discuss the path of a photon or an electron, you might suggest that we simply measure the particle's position and velocity many times in a row. This series of snapshots would amount to a description of its path.

A practical objection to this plan is that the process of measurement will have an effect on the thing we are trying to measure. This may not be of much concern, for example, when a traffic cop measures your car's motion with a radar gun, because the energy and momentum of the radar pulses aren't enough to change the car's motion significantly. But on the subatomic scale it is a very real problem. Making a videotape of an electron orbiting a nucleus is not just difficult, it is theoretically impossible, even with the video camera hooked up to the best imaginable microscope. The video camera makes pictures of things using light that has bounced off them and come into the camera. If even a single photon of the right wavelength was to bounce off of the electron we were trying to study, the electron's recoil would be enough to change its behavior significantly (see homework problem 4).

The Heisenberg uncertainty principle

This insight, that measurement changes the thing being measured, is the kind of idea that clove-cigarette-smoking intellectuals outside of the physical sciences like to claim they knew all along. If only, they say, the physicists had made more of a habit of reading literary journals, they could have saved a lot of work. The anthropologist Margaret Mead has recently been accused of inadvertently encouraging her teenaged Samoan informants to exaggerate the freedom of youthful sexual experimentation in their society. If this is considered a damning critique of her work, it is because she could have done better: other anthropologists claim to have been able to eliminate the observer-as-participant problem and collect untainted data.

The German physicist Werner Heisenberg, however, showed that in quantum physics, *any* measuring technique runs into a brick wall when we try to improve its accuracy beyond a certain point. Heisenberg showed that the limitation is a question of *what there is to be known*, even in principle, about the system itself, not of the inability of a particular measuring device to ferret out information that is knowable.

Suppose, for example, that we have constructed an electron in a box (quantum dot) setup in our laboratory, and we are able to adjust the length L of the box as desired. All the standing wave patterns pretty much fill the box, so our knowledge of the electron's position is of limited accuracy. If we write Δx for the range of uncertainty in our knowledge of its position, then Δx is roughly the same as the length of the box:

$$\Delta x \approx L$$

If we wish to know its position more accurately, we can certainly squeeze it into a smaller space by reducing L, but this has an unintended side-effect. A standing wave is really a superposition of two traveling waves going in opposite directions. The equation $p = h/\lambda$ only gives the magnitude of the momentum vector, not its direction, so we should really interpret the wave as a 50/50 mixture of a right-going wave with momentum $p = h/\lambda$ and a left-going one with momentum $p = -h/\lambda$. The uncertainty in our knowledge of the electron's momentum is $\Delta p = 2h/\lambda$, covering the range between these two values. Even if we make sure the electron is in the ground state, whose wavelength $\lambda = 2L$ is the longest possible, we have an uncertainty in momentum of $\Delta p = h/L$. In general, we find

$$\Delta p \gtrsim h/L \qquad ,$$

with equality for the ground state and inequality for the higher-energy states. Thus if we reduce L to improve our knowledge of the electron's position, we do so at the cost of knowing less about its momentum. This trade-off is neatly summarized by multiplying the two equations to give

$$\Delta p \Delta x \gtrsim h \qquad .$$

Although we have derived this in the special case of a particle in a box, it is an example of a principle of more general validity:

the Heisenberg uncertainty principle
It is not possible, even in principle, to know the momentum and the position of a particle simultaneously and with perfect accuracy. The uncertainties in these two quantities are always such that

$$\Delta p \Delta x \gtrsim h \qquad .$$

(This approximation can be made into a strict inequality, $\Delta p \Delta x > h/4\pi$, but only with more careful definitions, which we will not bother with.[1])

Note that although I encouraged you to think of this derivation in terms of a specific real-world system, the quantum dot, I never

[1] See homework problems 6 and 7.

made any reference to specific measuring equipment. The argument is simply that we cannot *know* the particle's position very accurately unless it *has* a very well defined position, it cannot have a very well defined position unless its wave-pattern covers only a very small amount of space, and its wave-pattern cannot be thus compressed without giving it a short wavelength and a correspondingly uncertain momentum. The uncertainty principle is therefore a restriction on how much there is to know about a particle, not just on what we can know about it with a certain technique.

An estimate for electrons in atoms *example 6*

▷ A typical energy for an electron in an atom is on the order of (1 volt)$\cdot e$, which corresponds to a speed of about 1% of the speed of light. If a typical atom has a size on the order of 0.1 nm, how close are the electrons to the limit imposed by the uncertainty principle?

▷ If we assume the electron moves in all directions with equal probability, the uncertainty in its momentum is roughly twice its typical momentum. This only an order-of-magnitude estimate, so we take Δp to be the same as a typical momentum:

$$\begin{aligned}\Delta p \Delta x &= p_{typical}\Delta x \\ &= (m_{electron})(0.01c)(0.1\times10^{-9}\ \text{m}) \\ &= 3\times10^{-34}\ \text{J}\cdot\text{s}\end{aligned}$$

This is on the same order of magnitude as Planck's constant, so evidently the electron is "right up against the wall." (The fact that it is somewhat less than h is of no concern since this was only an estimate, and we have not stated the uncertainty principle in its most exact form.)

self-check B

If we were to apply the uncertainty principle to human-scale objects, what would be the significance of the small numerical value of Planck's constant? ▷ Answer, p. 1012

self-check C

Suppose rain is falling on your roof, and there is a tiny hole that lets raindrops into your living room now and then. All these drops hit the same spot on the floor, so they have the same value of x. Not only that, but if the rain is falling straight down, they all have zero horizontal momentum. Thus it seems that the raindrops have $\Delta p = 0$, $\Delta x = 0$, and $\Delta p\Delta x = 0$, violating the uncertainty principle. To look for the hole in this argument, consider how it would be acted out on the microscopic scale: an electron wave comes along and hits a narrow slit. What really happens? ▷ Answer, p. 1012

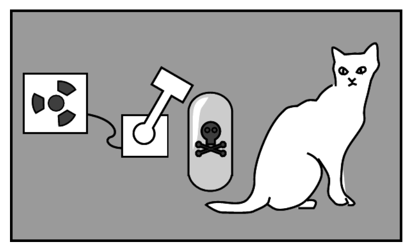
Measurement and Schrödinger's cat

In chapter 34 I briefly mentioned an issue concerning measurement that we are now ready to address carefully. If you hang around a laboratory where quantum-physics experiments are being done and secretly record the physicists' conversations, you'll hear them say many things that assume the probability interpretation of quantum mechanics. Usually they will speak as though the randomness of quantum mechanics enters the picture when something is measured. In the digital camera experiments of chapter 34, for example, they would casually describe the detection of a photon at one of the pixels as if the moment of detection was when the photon was forced to "make up its mind." Although this mental cartoon usually works fairly well as a description of things one experiences in the lab, it cannot ultimately be correct, because it attributes a special role to measurement, which is really just a physical process like all other physical processes.[2]

If we are to find an interpretation that avoids giving any special role to measurement processes, then we must think of the entire laboratory, including the measuring devices and the physicists themselves, as one big quantum-mechanical system made out of protons, neutrons, electrons, and photons. In other words, we should take quantum physics seriously as a description not just of microscopic objects like atoms but of human-scale ("macroscopic") things like the apparatus, the furniture, and the people.

The most celebrated example is called the Schrödinger's cat experiment. Luckily for the cat, there probably was no actual experiment — it was simply a "thought experiment" that the German theorist Schrödinger discussed with his colleagues. Schrödinger wrote:

> One can even construct quite burlesque cases. A cat is shut up in a steel container, together with the following diabolical apparatus (which one must keep out of the direct clutches of the cat): In a [radiation detector] there is a tiny mass of radioactive substance, so little that in the course of an hour perhaps one atom of it disintegrates, but also with equal probability not even one; if it does happen, the [detector] responds and ... activates a hammer that shatters a little flask of prussic acid [filling the chamber with poison gas]. If one has left this entire system to itself for an hour, then one will say to himself that the cat is still living, if in that time no atom has disintegrated. The first atomic disintegration would have poisoned it.



j / Schrödinger's cat.

Now comes the strange part. Quantum mechanics says that the particles the cat is made of have wave properties, including the property of superposition. Schrödinger describes the wavefunction of the box's contents at the end of the hour:

> The wavefunction of the entire system would express this situation by having the living and the dead cat mixed ... in equal parts [50/50 proportions]. The uncertainty originally restricted to the atomic domain has been transformed into a macroscopic uncertainty...

[2]This interpretation of quantum mechanics is called the Copenhagen interpretation, because it was originally developed by a school of physicists centered in Copenhagen and led by Niels Bohr.

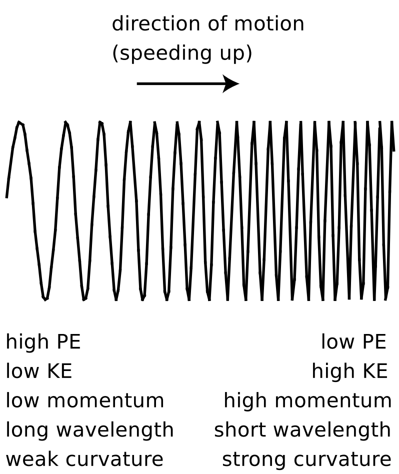
At first Schrödinger's description seems like nonsense. When you opened the box, would you see two ghostlike cats, as in a doubly exposed photograph, one dead and one alive? Obviously not. You would have a single, fully material cat, which would either be dead or very, very upset. But Schrödinger has an equally strange and logical answer for that objection. In the same way that the quantum randomness of the radioactive atom spread to the cat and made its wavefunction a random mixture of life and death, the randomness spreads wider once you open the box, and your own wavefunction becomes a mixture of a person who has just killed a cat and a person who hasn't.[3]

Discussion questions

A Compare Δp and Δx for the two lowest energy levels of the one-dimensional particle in a box, and discuss how this relates to the uncertainty principle.

B On a graph of Δp versus Δx, sketch the regions that are allowed and forbidden by the Heisenberg uncertainty principle. Interpret the graph: Where does an atom lie on it? An elephant? Can either p or x be measured with perfect accuracy if we don't care about the other?

35.5 Electrons in electric fields



k / An electron in a gentle electric field gradually shortens its wavelength as it gains energy.

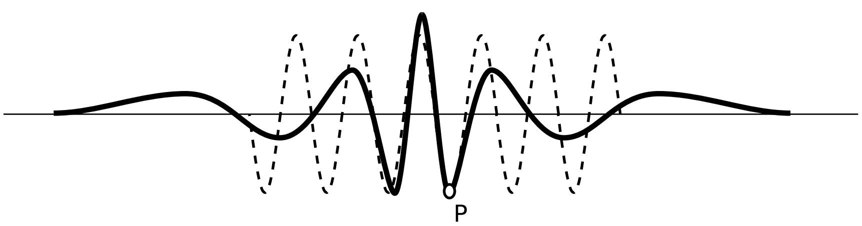
So far the only electron wave patterns we've considered have been simple sine waves, but whenever an electron finds itself in an electric field, it must have a more complicated wave pattern. Let's consider the example of an electron being accelerated by the electron gun at the back of a TV tube. The electron is moving from a region of low voltage into a region of higher voltage. Since its charge is negative, it loses PE by moving to a higher voltage, so its KE increases. As its potential energy goes down, its kinetic energy goes up by an equal amount, keeping the total energy constant. Increasing kinetic energy implies a growing momentum, and therefore a shortening wavelength, k.

The wavefunction as a whole does not have a single well-defined wavelength, but the wave changes so gradually that if you only look at a small part of it you can still pick out a wavelength and relate it to the momentum and energy. (The picture actually exaggerates by many orders of magnitude the rate at which the wavelength changes.)

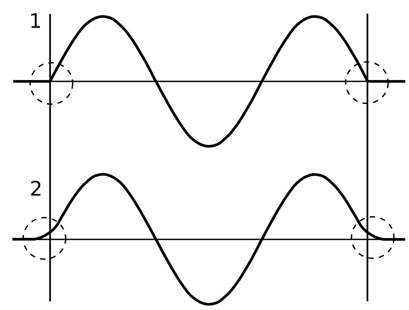
But what if the electric field was stronger? The electric field in a TV is only $\sim 10^5$ N/C, but the electric field within an atom is more like 10^{12} N/C. In figure l, the wavelength changes so rapidly that there is nothing that looks like a sine wave at all. We could get a general idea of the wavelength in a given region by measuring the distance

[3] This interpretation, known as the many-worlds interpretation, was developed by Hugh Everett in 1957.

between two peaks, but that would only be a rough approximation. Suppose we want to know the wavelength at point P. The trick is to construct a sine wave, like the one shown with the dashed line, which matches the curvature of the actual wavefunction as closely as possible near P. The sine wave that matches as well as possible is called the "osculating" curve, from a Latin word meaning "to kiss." The wavelength of the osculating curve is the wavelength that will relate correctly to conservation of energy.



l / A typical wavefunction of an electron in an atom (heavy curve) and the osculating sine wave (dashed curve) that matches its curvature at point P.



m / 1. Kinks like this don't happen. 2. The wave actually penetrates into the classically forbidden region.

Tunneling

We implicitly assumed that the particle-in-a-box wavefunction would cut off abruptly at the sides of the box, m/1, but that would be unphysical. A kink has infinite curvature, and curvature is related to energy, so it can't be infinite. A physically realistic wavefunction must always "tail off" gradually, m/2. In classical physics, a particle can never enter a region in which its potential energy would be greater than the amount of energy it has available. But in quantum physics the wavefunction will always have a tail that reaches into the classically forbidden region. If it was not for this effect, called tunneling, the fusion reactions that power the sun would not occur due to the high potential energy that nuclei need in order to get close together! Tunneling is discussed in more detail in the next section.

35.6 $\int \star$ The Schrödinger equation

In section 35.5 we were able to apply conservation of energy to an electron's wavefunction, but only by using the clumsy graphical technique of osculating sine waves as a measure of the wave's curvature. You have learned a more convenient measure of curvature in calculus: the second derivative. To relate the two approaches, we take the second derivative of a sine wave:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin\left(\frac{2\pi x}{\lambda}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2\pi}{\lambda}\cos\frac{2\pi x}{\lambda}\right)$$

$$= -\left(\frac{2\pi}{\lambda}\right)^2 \sin\frac{2\pi x}{\lambda}$$

Taking the second derivative gives us back the same function, but with a minus sign and a constant out in front that is related to the wavelength. We can thus relate the second derivative to the osculating wavelength:

[1] $$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} = -\left(\frac{2\pi}{\lambda}\right)^2\Psi$$

This could be solved for λ in terms of Ψ, but it will turn out to be more convenient to leave it in this form.

Using conservation of energy, we have

[2] $$\begin{aligned} E &= KE + PE \\ &= \frac{p^2}{2m} + PE \\ &= \frac{(h/\lambda)^2}{2m} + PE \end{aligned}$$

Note that both equation [1] and equation [2] have λ^2 in the denominator. We can simplify our algebra by multiplying both sides of equation [2] by Ψ to make it look more like equation [1]:

$$\begin{aligned} E\cdot\Psi &= \frac{(h/\lambda)^2}{2m}\Psi + PE\cdot\Psi \\ &= \frac{1}{2m}\left(\frac{h}{2\pi}\right)^2\left(\frac{2\pi}{\lambda}\right)^2\Psi + PE\cdot\Psi \\ &= -\frac{1}{2m}\left(\frac{h}{2\pi}\right)^2\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + PE\cdot\Psi \end{aligned}$$

Further simplification is achieved by using the symbol $\hbar$ (h with a slash through it, read "h-bar") as an abbreviation for $h/2\pi$. We then have the important result known as the **Schrödinger equation**:

$$E\cdot\Psi = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + PE\cdot\Psi$$

(Actually this is a simplified version of the Schrödinger equation, applying only to standing waves in one dimension.) Physically it is a statement of conservation of energy. The total energy E must be constant, so the equation tells us that a change in potential energy must be accompanied by a change in the curvature of the wavefunction. This change in curvature relates to a change in wavelength, which corresponds to a change in momentum and kinetic energy.

self-check D

Considering the assumptions that were made in deriving the Schrödinger equation, would it be correct to apply it to a photon? To an electron moving at relativistic speeds? ▷ Answer, p. 1012

Usually we know right off the bat how the potential energy depends on x, so the basic mathematical problem of quantum physics is to find a function $\Psi(x)$ that satisfies the Schrödinger equation for a given function $PE(x)$. An equation, such as the Schrödinger equation, that specifies a relationship between a function and its derivatives is known as a differential equation.

The study of differential equations in general is beyond the mathematical level of this book, but we can gain some important insights by considering the easiest version of the Schrödinger equation, in which the potential energy is constant. We can then rearrange the Schrödinger equation as follows:

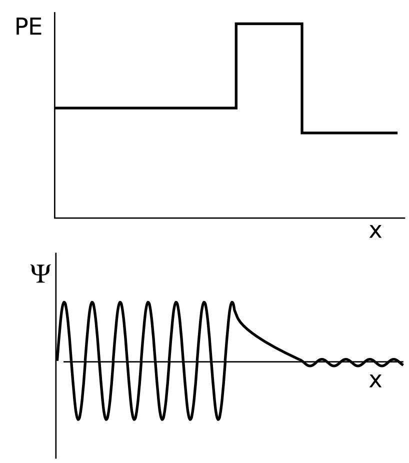
$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} = \frac{2m(PE-E)}{\hbar^2}\Psi \qquad ,$$

which boils down to

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} = a\Psi \qquad ,$$

where, according to our assumptions, a is independent of x. We need to find a function whose second derivative is the same as the original function except for a multiplicative constant. The only functions with this property are sine waves and exponentials:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\,q\sin(rx+s)\,\right] = -qr^2\sin(rx+s)$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[qe^{rx+s}\right] = qr^2e^{rx+s}$$



n / Tunneling through a barrier.

The sine wave gives negative values of a, $a=-r^2$, and the exponential gives positive ones, $a=r^2$. The former applies to the classically allowed region with $PE<E$.

This leads us to a quantitative calculation of the tunneling effect discussed briefly in the preceding subsection. The wavefunction evidently tails off exponentially in the classically forbidden region. Suppose, as shown in figure n, a wave-particle traveling to the right encounters a barrier that it is classically forbidden to enter. Although the form of the Schrödinger equation we're using technically does not apply to traveling waves (because it makes no reference to time), it turns out that we can still use it to make a reasonable calculation of the probability that the particle will make it through the barrier. If we let the barrier's width be w, then the ratio of the wavefunction on the left side of the barrier to the wavefunction on the right is

$$\frac{qe^{rx+s}}{qe^{r(x+w)+s}} = e^{-rw} \qquad .$$

Probabilities are proportional to the squares of wavefunctions, so the probability of making it through the barrier is

$$\begin{aligned} P &= e^{-2rw} \\ &= \exp\left(-\frac{2w}{\hbar}\sqrt{2m(PE-E)}\right) \end{aligned}$$

self-check E

If we were to apply this equation to find the probability that a person can walk through a wall, what would the small value of Planck's constant imply? ▷ Answer, p. 1012

Use of complex numbers

In a classically forbidden region, a particle's total energy, $PE+KE$, is less than its PE, so its KE must be negative. If we want to keep believing in the equation $KE = p^2/2m$, then apparently the momentum of the particle is the square root of a negative number. This is a symptom of the fact that the Schrödinger equation fails to describe all of nature unless the wavefunction and various other quantities are allowed to be complex numbers. In particular it is not possible to describe traveling waves correctly without using complex wavefunctions.

This may seem like nonsense, since real numbers are the only ones that are, well, real! Quantum mechanics can always be related to the real world, however, because its structure is such that the results of measurements always come out to be real numbers. For example, we may describe an electron as having non-real momentum in classically forbidden regions, but its average momentum will always come out to be real (the imaginary parts average out to zero), and it can never transfer a non-real quantity of momentum to another particle.

A complete investigation of these issues is beyond the scope of this book, and this is why we have normally limited ourselves to standing waves, which can be described with real-valued wavefunctions.

Summary

Selected vocabulary

wavefunction . .	the numerical measure of an electron wave, or in general of the wave corresponding to any quantum mechanical particle

Notation

$\hbar$	Planck's constant divided by 2π (used only in optional section 35.6)
Ψ	the wavefunction of an electron

Summary

Light is both a particle and a wave. Matter is both a particle and a wave. The equations that connect the particle and wave properties are the same in all cases:

$$E = hf$$
$$p = h/\lambda$$

Unlike the electric and magnetic fields that make up a photon-wave, the electron wavefunction is not directly measurable. Only the square of the wavefunction, which relates to probability, has direct physical significance.

A particle that is bound within a certain region of space is a standing wave in terms of quantum physics. The two equations above can then be applied to the standing wave to yield some important general observations about bound particles:

1. The particle's energy is quantized (can only have certain values).
2. The particle has a minimum energy.
3. The smaller the space in which the particle is confined, the higher its kinetic energy must be.

These immediately resolve the difficulties that classical physics had encountered in explaining observations such as the discrete spectra of atoms, the fact that atoms don't collapse by radiating away their energy, and the formation of chemical bonds.

A standing wave confined to a small space must have a short wavelength, which corresponds to a large momentum in quantum physics. Since a standing wave consists of a superposition of two traveling waves moving in opposite directions, this large momentum should actually be interpreted as an equal mixture of two possible momenta: a large momentum to the left, or a large momentum to the right. Thus it is not possible for a quantum wave-particle to be confined to a small space without making its momentum very uncertain. In general, the Heisenberg uncertainty principle states that it is not possible to know the position and momentum of a particle simultaneously with perfect accuracy. The uncertainties in these two quantities must satisfy the approximate inequality

$$\Delta p \Delta x \gtrsim h \qquad .$$

When an electron is subjected to electric forces, its wavelength cannot be constant. The "wavelength" to be used in the equation $p = h/\lambda$ should be thought of as the wavelength of the sine wave that most closely approximates the curvature of the wavefunction at a specific point.

Infinite curvature is not physically possible, so realistic wavefunctions cannot have kinks in them, and cannot just cut off abruptly at the edge of a region where the particle's energy would be insufficient to penetrate according to classical physics. Instead, the wavefunction "tails off" in the classically forbidden region, and as a consequence it is possible for particles to "tunnel" through regions where according to classical physics they should not be able to penetrate. If this quantum tunneling effect did not exist, there would be no fusion reactions to power our sun, because the energies of the nuclei would be insufficient to overcome the electrical repulsion between them.

Exploring further

The New World of Mr. Tompkins: George Gamow's Classic Mr. Tompkins in Paperback, George Gamow. Mr. Tompkins finds himself in a world where the speed of light is only 30 miles per hour, making relativistic effects obvious. Later parts of the book play similar games with Planck's constant.

The First Three Minutes: A Modern View of the Origin of the Universe, Steven Weinberg. Surprisingly simple ideas allow us to understand the infancy of the universe surprisingly well.

Three Roads to Quantum Gravity, Lee Smolin. The greatest embarrassment of physics today is that we are unable to fully reconcile general relativity (the theory of gravity) with quantum mechanics. This book does a good job of introducing the lay reader to a difficult, speculative subject, and showing that even though we don't have a full theory of quantum gravity, we do have a clear outline of what such a theory must look like.

Problems

Key

√ A computerized answer check is available online.
∫ A problem that requires calculus.
⋆ A difficult problem.

1 In a television, suppose the electrons are accelerated from rest through a voltage difference of 10^4 V. What is their final wavelength? √

2 Use the Heisenberg uncertainty principle to estimate the minimum velocity of a proton or neutron in a ^{208}Pb nucleus, which has a diameter of about 13 fm (1 fm = 10^{-15} m). Assume that the speed is nonrelativistic, and then check at the end whether this assumption was warranted. √

3 A free electron that contributes to the current in an ohmic material typically has a speed of 10^5 m/s (much greater than the drift velocity).
(a) Estimate its de Broglie wavelength, in nm. √
(b) If a computer memory chip contains 10^8 electric circuits in a 1 cm^2 area, estimate the linear size, in nm, of one such circuit. √
(c) Based on your answers from parts a and b, does an electrical engineer designing such a chip need to worry about wave effects such as diffraction?
(d) Estimate the maximum number of electric circuits that can fit on a 1 cm^2 computer chip before quantum-mechanical effects become important.

4 On page 968, I discussed the idea of hooking up a video camera to a visible-light microscope and recording the trajectory of an electron orbiting a nucleus. An electron in an atom typically has a speed of about 1% of the speed of light.
(a) Calculate the momentum of the electron. √
(b) When we make images with photons, we can't resolve details that are smaller than the photons' wavelength. Suppose we wanted to map out the trajectory of the electron with an accuracy of 0.01 nm. What part of the electromagnetic spectrum would we have to use?
(c) As found in homework problem 12 on page 787, the momentum of a photon is given by $p = E/c$. Estimate the momentum of a photon having the necessary wavelength. √
(d) Comparing your answers from parts a and c, what would be the effect on the electron if the photon bounced off of it? What does this tell you about the possibility of mapping out an electron's orbit around a nucleus?

5 Find the energy of a particle in a one-dimensional box of length L, expressing your result in terms of L, the particle's mass m, the number of peaks and valleys n in the wavefunction, and fundamental constants. ✓

6 The Heisenberg uncertainty principle, $\Delta p \Delta x \gtrsim h$, can only be made into a strict inequality if we agree on a rigorous mathematical definition of Δx and Δp. Suppose we define the deltas in terms of the full width at half maximum (FWHM), which we first encountered on p. 467 and revisited on page 923 of this book. Now consider the lowest-energy state of the one-dimensional particle in a box. As argued on page 969, the momentum has equal probability of being h/L or $-h/L$, so the FWHM definition gives $\Delta p = 2h/L$.
(a) Find Δx using the FWHM definition. Keep in mind that the probability distribution depends on the square of the wavefunction.
(b) Find $\Delta x \Delta p$. ✓

7 If x has an average value of zero, then the standard deviation of the probability distribution $D(x)$ is defined by

$$\sigma^2 = \sqrt{\int D(x) x^2 \, \mathrm{d}x} \qquad ,$$

where the integral ranges over all possible values of x.

Interpretation: if x only has a high probability of having values close to the average (i.e., small positive and negative values), the thing being integrated will always be small, because x^2 is always a small number; the standard deviation will therefore be small. Squaring x makes sure that either a number below the average ($x < 0$) or a number above the average ($x > 0$) will contribute a positive amount to the standard deviation. We take the square root of the whole thing so that it will have the same units as x, rather than having units of x^2.

Redo problem 6 using the standard deviation rather than the FWHM.

Hints: (1) You need to determine the amplitude of the wave based on normalization. (2) You'll need the following definite integral: $\int_{-\pi/2}^{\pi/2} u^2 \cos^2 u \, \mathrm{d}u = (\pi^3 - 6\pi)/24$. ✓ $\int$

8 In section 35.6 we derived an expression for the probability that a particle would tunnel through a rectangular potential barrier. Generalize this to a barrier of any shape. [Hints: First try generalizing to two rectangular barriers in a row, and then use a series of rectangular barriers to approximate the actual curve of an arbitrary potential. Note that the width and height of the barrier in the original equation occur in such a way that all that matters is the area under the PE-versus-x curve. Show that this is still true for a series of rectangular barriers, and generalize using an integral.] If you had done this calculation in the 1930's you could have become a famous physicist. $\int$ ⋆

9 The electron, proton, and neutron were discovered, respectively, in 1897, 1919, and 1932. The neutron was late to the party, and some physicists felt that it was unnecessary to consider it as fundamental. Maybe it could be explained as simply a proton with an electron trapped inside it. The charges would cancel out, giving the composite particle the correct neutral charge, and the masses at least approximately made sense (a neutron is heavier than a proton). (a) Given that the diameter of a proton is on the order of 10^{-15} m, use the Heisenberg uncertainty principle to estimate the trapped electron's minimum momentum. $\checkmark$
(b) Find the electron's minimum kinetic energy. $\checkmark$
(c) Show via $E = mc^2$ that the proposed explanation fails, because the contribution to the neutron's mass from the electron's kinetic energy would be many orders of magnitude too large.